Knot arithmetic. Conway polynomial

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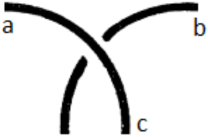
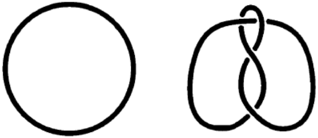
n-coloring



2c=a+b (mod n)

Can we distinguish figure 8 knot from unknot?



First method

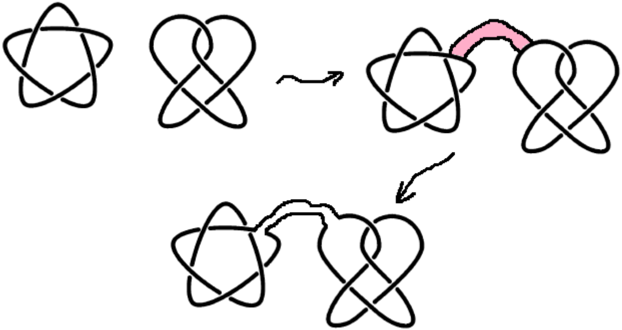
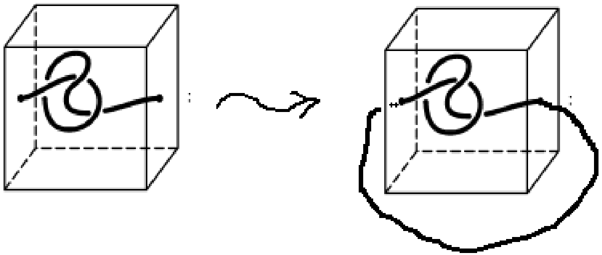


Figure : Connected product of two knots

Box description of a knot



Second method

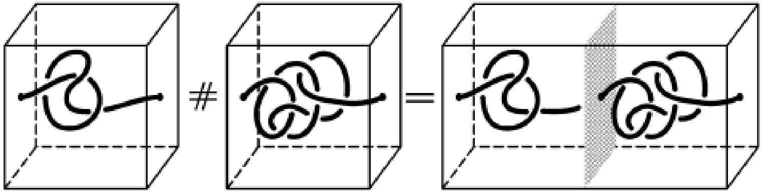


Figure : Connected product in box presentation

Properties

Lemma

The operation $\#$ is associative and commutative, and the unknot U is the identity element. In other words, for any three knots A, B, C, we have $(A\#B)\#C = A\#(B\#C)$, $A\#B = B\#A$ and $U\#A = A\#U = A$. So we can denote the unknot U simply by 1.

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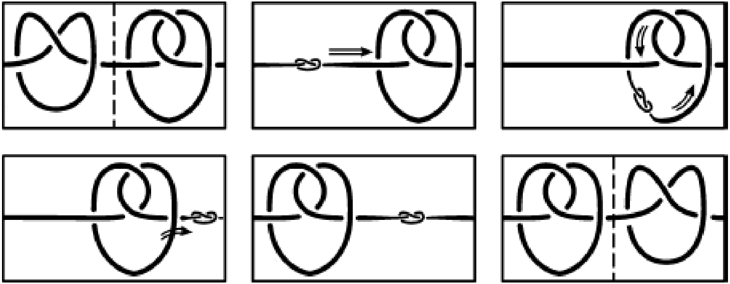

Figure : Commutativity of composition

Knots don't have inverses

Theorem

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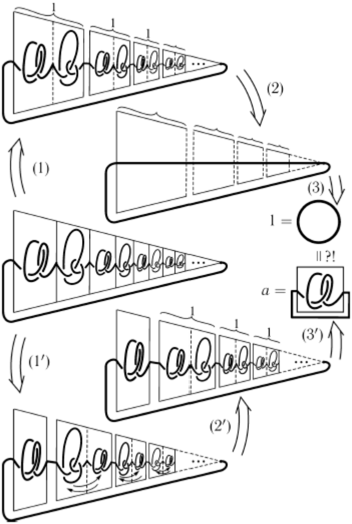


n-coloring of a connected sum

Theorem

$col_n(A) \cdot col_n(B) = n \cdot col_n(A\#B)$.

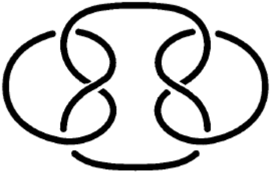
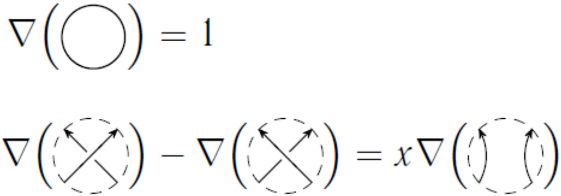


Figure : Connected sum of two trefoils

Defining relations

Definition

A Conway polynomial of a link is a function ∇ giving for any diagram D a polynomial $\nabla(D)$ in one variable x defined by the following two relations, called skein relations:



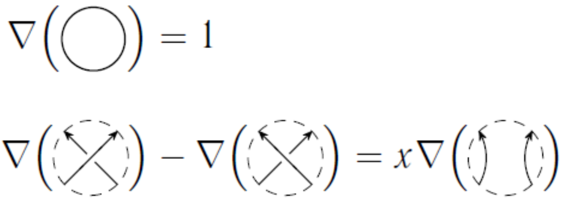
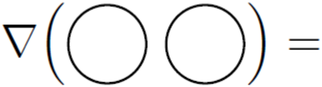


Figure : Conway relations



Examples

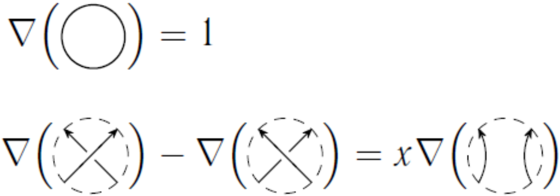
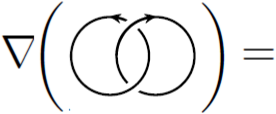
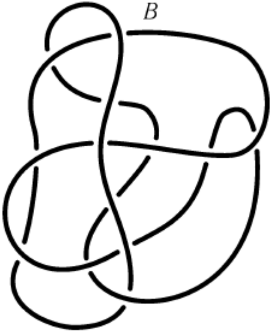


Figure : Conway relations



$$\nabla(A) = 1 - x^2$$



$$\nabla(B) = x^2 + 1$$